

Backstepping Control of the One-Phase Stefan Problem

Shumon Koga, Mamadou Diagne, Shuxia Tang, and Miroslav Krstic

Abstract—In this paper, a backstepping control of the one-phase Stefan Problem, which is a 1-D diffusion Partial Differential Equation (PDE) defined on a time varying spatial domain described by an ordinary differential equation (ODE), is studied. A new nonlinear backstepping transformation for moving boundary problem is utilized to transform the original coupled PDE-ODE system into a target system whose exponential stability is proved. The full-state boundary feedback controller ensures the exponential stability of the moving interface to a reference setpoint and the \mathcal{H}_1 -norm of the distributed temperature by a choice of the setpoint satisfying given explicit inequality between initial states that guarantees the physical constraints imposed by the melting process.

I. INTRODUCTION

Diffusion PDEs with moving boundaries have been studied actively for the last few decades, and their understanding continues to be of high interest due to their extent for various industrial processes. Representative applications include sea-ice melting and freezing [1], continuous casting of steel [2], crystal-growth [3], and thermal energy storage system [4].

While the numerical analysis of these systems is widely covered in the literature, their control related problems have been addressed relatively fewer. In addition to it, most of the proposed control approaches are based on finite dimensional approximations with the assumption of an explicit and known moving boundary [5], [6], [7]. For instance, diffusion-reaction processes with explicitly known moving boundaries are investigated in [6] based on the concept of inertial manifold [8] and the partitioning of the infinite dimensional dynamics into slow and fast finite dimensional modes. We also refer the reader to [7] for the motion planning boundary control of a one-dimensional one-phase nonlinear Stefan problem based on series representation which leads to highly complex solutions that reduce the design possibilities. Instead of controlling the position of the liquid-solid interface by means of the temperature or the heat flux at the boundary, [7] solves the inverse problem assuming a prior known moving boundary which help to derive the manipulated input.

More complicated approaches that lead to significant mathematical complexities in the process characterization are developed based on an infinite dimensional framework. These methods impose a number of constraints on the initial data and the state variables to achieve convergence of the

dynamical coupled systems to the desired equilibrium. In order to avoid surface cracks during the solidification stage in a steel casting process represented by a diffusion PDE-ODE system defined on a time-varying spatial domain, an enthalpy-based Lyapunov functional is used in [2] to ensure stabilization of the temperature and the moving boundary at the desired setpoint. The aftermentioned results are derived restricting a priori, the input signal to be strictly positive and the moving boundary to be a non-decreasing function of time. In [9] a geometric control approach [10]–[12] that enables the manipulation of the boundary heat flux to adjust the position of a liquid-solid interface at a desired setpoint is proposed, and the exponential stability of the \mathcal{L}_2 -norm of the distributed temperature is ensured using a Lyapunov analysis.

In this paper, the backstepping control [13], [14] of a one phase Stefan problem [2], [9], [15] is studied. The exploitation of the PDE backstepping methodology for moving boundary problems is not well investigated in the literature. An extension of the backstepping-based observer design to the state estimation of parabolic PDEs with time-dependent spatial domain was proposed in [16] with an application to the well-known Czochralski crystal growth process. The authors offer a design tool for the stabilization of an unstable parabolic PDE system with moving interface using a collocated boundary measurement and actuation located at the moving boundary. The novelty of our approach relies to the proposition of a new nonlinear backstepping transformation that allows to deal with moving boundary problem without the need to rescale the system into a fixed domain. Our design of backstepping transformation for moving boundary stands as an extension of the one proposed in [17]–[19] for linear PDE-ODE defined on a fixed spatial domain. The proposed controller enables the exponential stability of sum of the moving interface and the \mathcal{H}_1 -norm of the temperature profile under physical constraints which restrict the choice of reference setpoint with respect to the initial data.

This paper is organized as follows: The one-phase Stefan problem is presented in Section II and the control related problem in Section III. Section IV introduces the nonlinear backstepping transformation for moving boundary problems. The properties of the proposed backstepping controller are stated in Section V and the Lyapunov stability of the closed-loop system under full-state feedback is established in Section VI with the exponential convergence of \mathcal{H}_1 -norm of the distributed temperature and the moving boundary to the desired equilibrium. Supportive numerical simulations are provided in Section VII. The paper ends with final remarks and future directions discussed in Section VIII.

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II. DESCRIPTION OF THE PHYSICAL PROCESS

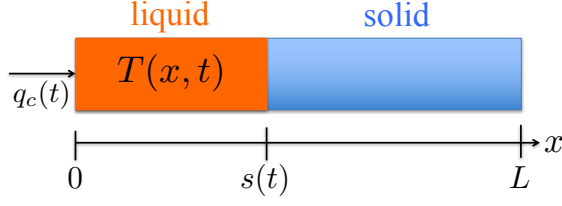


Fig. 1. Schematic of 1D Stefan problem.

Consider a physical model which describes the melting or solidification mechanism in a pure one-component material of length L in one dimension. In order to describe a position at which phase transition from liquid to solid occurs (or equivalently, in the reverse direction) mathematically, we divide the domain $[0, L]$ into the two time varying sub-domains, namely, $[0, s(t)]$ occupied by the liquid phase, and $[s(t), L]$ by the solid phase. A heat flux is manipulated into the material through the boundary at $x = 0$ of the liquid phase, which affects the dynamics of the solid-liquid interface. As a consequence, the heat equation alone does not provide a complete description of the phase transition and must be coupled with an unknown dynamics that describes the moving boundary. Assuming that the temperature in the liquid phase is not lower than the melting temperature T_m of the material, the following coupled system consisting of

- the diffusion equation of the temperature in the liquid-phase which is written as

$$T_t(x, t) = \alpha T_{xx}(x, t), \quad 0 \leq x \leq s(t), \quad \alpha := \frac{k}{\rho C_p}, \quad (1)$$

with the boundary conditions

$$-kT_x(0, t) = q_c(t), \quad (2)$$

$$T(s(t), t) = T_m, \quad (3)$$

and the initial conditions

$$T(x, 0) = T_0(x), \quad s_0 = s_0 \quad (4)$$

where $T(x, t)$, $q_c(t)$, ρ , C_p and k are the distributed temperature of the liquid phase, manipulated heat flux, liquid density, the liquid heat capacity, and the liquid heat conductivity, respectively.

and

- the local energy balance at the position of the liquid-solid interface $x = s(t)$ which yields to the following ODE

$$\dot{s}(t) = -\beta T_x(s(t), t), \quad \beta := \frac{k}{\rho \Delta H^*} \quad (5)$$

that describes the dynamics of moving boundary. The physical parameter ΔH^* represents the latent heat of fusion.

For the sake of brevity, we refer the readers to [20], where the Stefan condition in the case of a solidification process is derived.

Remark 1: As the moving interface $s(t)$ is unknown explicitly, the problem defined in (1)–(5) is a highly nonlinear problem. Note that this non-linearity is purely geometric rather than algebraic.

Remark 2: Due to the so-called isothermal interface condition that prescribes the melting temperature T_m at the interface through (3), this form of the Stefan problem is a reasonable model only if the following conditions hold:

$$T(x, t) \geq T_m \quad \text{for all } x \in [0, s(t)], \quad (6)$$

$$\dot{s}(t) \geq 0 \quad \text{for all } t \geq 0 \quad (7)$$

From Remark 2, it is plausible to assume the existence of a positive constant H such that

$$0 \leq T_0(x) - T_m \leq H(s_0 - x). \quad (8)$$

We recall the following lemma that ensures the validity of the model (1)–(5).

Lemma 1: For any $q_c(t) > 0$ on the finite time interval $(0, \bar{t})$, $T(x, t) > T_m$, $\forall x \in (0, s(t))$ and $\forall t \in (0, \bar{t})$. And then $\dot{s}(t) > 0$, $\forall t \in (0, \bar{t})$.

The proof of the lemma 1 is based on Maximum Principle and Hopf's Lemma as shown in [20].

III. CONTROL PROBLEM STATEMENT

In this model, the heat flux $q_c(t)$ is manipulated as a boundary controller. The objective of control is to drive the moving boundary $s(t)$ to a desired position s_r while ensuring the convergence of the \mathcal{H}_1 -norm of the temperature $T(x, t)$ in the liquid phase. We denote the reference error of liquid temperature and the moving interface as $u(x, t) = T(x, t) - T_m$ and $X(t) = s(t) - s_r$, respectively. The main theorem of this paper is stated as follows.

Theorem 1: Consider a closed-loop system consisting of the plant (1)–(5) and the control law

$$q_c(t) = -ck \left(\frac{1}{\alpha} \int_0^{s(t)} (T(x, t) - T_m) dx + \frac{1}{\beta} (s(t) - s_r) \right). \quad (9)$$

where $c > 0$ is an arbitral controller gain. Assume that the initial condition $(T_0(x), s_0)$ is compatible with the control law and satisfies (8). For any reference setpoint s_r which satisfies the following inequality

$$s_r > s_0 + \frac{C_p}{\Delta H^*} \int_0^{s_0} (T_0(x) - T_m) dx, \quad (10)$$

the closed-loop system is exponentially stable in the sense of the norm

$$\|T - T_m\|_{\mathcal{H}_1}^2 + (s(t) - s_r)^2. \quad (11)$$

The proof is to be given below.

IV. BACKSTEPPING TRANSFORMATION FOR MOVING BOUNDARY FORMULATION

A. Direct transformation

In this section, we introduce the following new backstepping transformation in order to achieve a target system that is exponentially stable.

$$w(x, t) = u(x, t) - \frac{c}{\alpha} \int_x^{s(t)} (y - x) u(y, t) dy + \frac{c}{\beta} (s(t) - x) X(t) \quad (12)$$

This transformation is an extension of the one first introduced in [17], to moving boundary problems proposed in [16]. By taking the derivative of (12) with respect to t and x respectively along the solution of (1)-(5) with the control law (9), we derive the following target system

$$w_t(x, t) = \alpha w_{xx}(x, t) + \frac{c}{\beta} \dot{s}(t) X(t) \quad (13)$$

with the boundary conditions which are given as

$$w(s(t), t) = 0 \quad (14)$$

$$w_x(0, t) = 0. \quad (15)$$

with the help of (12), the ODE (5) is rewritten as

$$\dot{X}(t) = -cX(t) - \beta w_x(s(t), t) \quad (16)$$

B. Inverse transformation

The original system (1)–(5) and the target system (13)–(16) have equivalent stability properties if the transformation (12) is invertible. Let us consider the following inverse transformation

$$u(x, t) = w(x, t) + \int_x^{s(t)} l(x, y) w(y, t) dy + h(s(t) - x) X(t), \quad (17)$$

where $l(x, y)$ is the kernel function. Taking derivative of (17) with respect to t and x respectively along the solution of (13)–(16), the following relation is derived

$$\begin{aligned} u_t(x, t) - \alpha u_{xx}(x, t) &= 2\alpha \left(\frac{d}{dx} l(x, x) \right) w(x, t) \\ &- \alpha \int_x^{s(t)} (l_{xx}(x, y) - l_{yy}(x, y)) w(y, t) dy \\ &+ \left(\frac{c}{\beta} \left(1 + \int_x^{s(t)} l(x, y) dy \right) + h'(s(t) - x) \right) \dot{s}(t) X(t) \\ &- (\alpha h''(s(t) - x) + ch(s(t) - x)) X(t) \\ &+ (\alpha l(x, s(t)) - \beta h(s(t) - x)) w_x(s(t), t) \end{aligned} \quad (18)$$

$$u(s(t), t) = h(0) X(t) \quad (19)$$

$$u_x(s(t), t) = w_x(s(t), t) - h'(0) X(t) \quad (20)$$

In order to hold (1)–(5) for any continuous functions $(w(x, t), X(t))$, by (18)–(20), $h(s(t) - x)$ and $l(x, y)$ satisfy

$$\alpha h''(s(t) - x) + ch(s(t) - x) = 0 \quad (21)$$

$$h(0) = 0, \quad h'(0) = -\frac{c}{\beta} \quad (22)$$

$$l_{xx}(x, y) - l_{yy}(x, y) = 0 \quad (23)$$

$$\frac{d}{dx} l(x, x) = 0 \quad (24)$$

$$\frac{c}{\beta} \left(1 + \int_x^{s(t)} l(x, y) dy \right) + h'(s(t) - x) = 0 \quad (25)$$

$$\alpha l(x, s(t)) - \beta h(s(t) - x) = 0 \quad (26)$$

By (21) and (22), the solution of $h(s(t) - x)$ is given by

$$h(s(t) - x) = -\frac{\sqrt{c\alpha}}{\beta} \sin \left(\sqrt{\frac{c}{\alpha}} (s(t) - x) \right) \quad (27)$$

In addition, the conditions (23)–(26) hold for

$$l(x, s(t)) = -\sqrt{\frac{c}{\alpha}} \sin \left(\sqrt{\frac{c}{\alpha}} (s(t) - x) \right) \quad (28)$$

Finally, from (27) and (28), the inverse transformation is written as

$$\begin{aligned} u(x, t) &= w(x, t) - \frac{\sqrt{c\alpha}}{\beta} \sin \left(\sqrt{\frac{c}{\alpha}} (s(t) - x) \right) X(t) \\ &- \int_x^{s(t)} \sqrt{\frac{c}{\alpha}} \sin \left(\sqrt{\frac{c}{\alpha}} (y - x) \right) w(y, t) dy \end{aligned} \quad (29)$$

V. PHYSICAL CONSTRAINTS

Noting that $q_c(t) > 0$ is required by Remark 2 and Lemma 1, the overshoot beyond the reference s_r is prohibited to achieve the control objective $s(t) \rightarrow s_r$ due to its increasing property (7), which means $s(t) < s_r$ is required to be satisfied for $\forall t > 0$. In this section, we derived the condition that guarantees these two conditions

$$q_c(t) > 0, \quad s(t) < s_r, \quad \forall t > 0 \quad (30)$$

, namely "physical constraints". Physically, a negative controller may lead to a freezing process.

Proposition 1: If the initial condition satisfies (10), then $q_c(t) > 0$ and $s_0 < s(t) < s_r$ for $\forall t > 0$.

Proof: By taking the time derivative of (9), we have

$$\begin{aligned} \dot{q}_c(t) &= -ck \left(\frac{1}{\alpha} \int_0^{s(t)} u_t(x, t) dx + \frac{1}{\alpha} \dot{s}(t) u(s(t), t) \right. \\ &\left. + \frac{1}{\beta} \dot{X}(t) \right) = ck u_x(0, t) = -cq_c(t) \end{aligned} \quad (31)$$

The solution of $q_c(t)$ can be written explicitly as

$$q_c(t) = q_c(0) e^{-ct} \quad (32)$$

Therefore, if the initial condition satisfies (10), which leads to $q_c(0) > 0$, then we can state that $q_c(t) > 0$ for any t . Next, by Lemma 1, if $q_c(t) > 0$ for $\forall t > 0$, then it deduces to

$$T(x, t) > T_m, \quad \dot{s}(t) > 0, \quad \forall y \in (0, s(t)), \forall t > 0. \quad (33)$$

Knowing $q_c(t) > 0$ and $T(x, t) > T_m$, by (9), we deduce

$$s(t) < s_r, \quad \forall t > 0 \quad (34)$$

In addition, by $\dot{s}(t) > 0$, it leads to $s_0 < s(t)$. Connecting this result with (34), we derive

$$s_0 < s(t) < s_r, \quad \forall t > 0. \quad (35)$$

In the following, we assume that (10) is satisfied. We use the property of (33) and (35) to show the Lyapunov stability. ■

VI. LYAPUNOV STABILITY

We recall that in order to design backstepping controller (9) and the target system (13)–(16) we introduce the transformation (12) and its inverse (17). The exponential stability of the scaled system (1)–(5) is guaranted if the nonlinear target system (13)–(16) is exponentially stable and the transformation (12) admits a unique inverse defined as (17). In the following we establish the exponential stability of the closed-loop control system H^1 -norm of the temperature and the moving boundary based on the analysis of the associated target system (13)–(16).

A. Exponential stability for the target (w, X) -system

Let V_1 be a functional such that

$$V_1 = \frac{1}{2} \int_0^{s(t)} w(x, t)^2 dx + \frac{1}{2} \int_0^{s(t)} w_x(x, t)^2 dx + \frac{p}{2} X(t)^2 \quad (36)$$

with a positive number $p > 0$ which is chosen later. Then, by taking the derivative of (36) along the solution of the target system (13)–(16), we have the following

$$\begin{aligned} \frac{dV_1}{dt} &= -\alpha \int_0^{s(t)} w_{xx}(x, t)^2 dx \\ &\quad - \alpha \int_0^{s(t)} w_x(x, t)^2 dx - pcX(t)^2 - p\beta X(t)w_x(s(t), t) \\ &\quad + \dot{s}(t) \left(\alpha \frac{c}{\beta} X(t) \int_0^{s(t)} w(x, t) dx \right. \\ &\quad \left. - \alpha \frac{c}{\beta} X(t)w_x(s(t), t) - \frac{1}{2} w_x(s(t), t)^2 \right) \end{aligned} \quad (37)$$

By (35), Pointcare's inequality and Agmon's inequality are obtained as

$$\int_0^{s(t)} w(x, t)^2 dx \leq 4s_r^2 \int_0^{s(t)} w_x(x, t)^2 dx, \quad (38)$$

$$w_x(s(t), t)^2 \leq 4s_r \int_0^{s(t)} w_{xx}(x, t)^2 dx. \quad (39)$$

Applying these, Young's inequality, Cauchy-Schwartz inequality, and (33) to (37), we have

$$\begin{aligned} \frac{dV_1}{dt} &\leq -\frac{\alpha}{4s_r^2 + 1} \left(\int_0^{s(t)} w_x(x, t)^2 dx + \int_0^{s(t)} w(x, t)^2 dx \right) \\ &\quad - p \left(c - \frac{p\beta^2 s_r}{\alpha} \right) X(t)^2 \\ &\quad + \frac{c\alpha \dot{s}(t)}{2\beta} \left(\int_0^{s(t)} w(x, t)^2 dx + \left(1 + \frac{c\alpha}{\beta} \right) X(t)^2 \right) \end{aligned} \quad (40)$$

Suppose that p is chosen to satisfy the following

$$0 < p < \frac{c\alpha}{\beta^2 s_r} \quad (41)$$

For any choice of $c > 0$, there exists $p > 0$ such that (41) holds. Consider the Lyapunov functional

$$V = V_1 e^{-as(t)} \quad (42)$$

then, by choosing the parameter a such that

$$a = \frac{c\alpha}{2\beta} \max \left\{ 1, \frac{1}{p} \left(1 + \frac{c\alpha}{\beta} \right) \right\}, \quad (43)$$

we have

$$\begin{aligned} \frac{dV}{dt} &\leq \left[-\frac{\alpha}{4s_r^2 + 1} \left(\int_0^{s(t)} w_x(x, t)^2 dx \right. \right. \\ &\quad \left. \left. + \int_0^{s(t)} w(x, t)^2 dx \right) - p \left(c - \frac{p\beta^2 s_r}{\alpha} \right) X(t)^2 \right] e^{-as(t)} \\ &\leq -bV \end{aligned} \quad (44)$$

where b is defined as

$$b = \min \left\{ \frac{2\alpha}{4s_r^2 + 1}, 2 \left(c - \frac{p\beta^2 s_r}{\alpha} \right) \right\} \quad (45)$$

By the relation (42) and Corollary 1 that $s_0 \leq s(t) \leq s_r$, we arrive at

$$V_1(t) \leq V_1(0) e^{a(s_r - s_0)} e^{-bt} \quad (46)$$

B. Exponential stability for the original (u, X) -system

From the direct transformation (12) and the inverse transformation (17) and using Young's and Cauchy-Schwarz inequality, we have

$$\int_0^{s(t)} w(x, t)^2 dx \leq M_1 \int_0^{s(t)} u(x, t)^2 dx + M_2 X(t)^2 \quad (47)$$

and

$$\begin{aligned} \int_0^{s(t)} w_x(x, t)^2 dx &\leq 3 \int_0^{s(t)} u_x(x, t)^2 dx \\ &\quad + M_3 \int_0^{s(t)} u(x, t)^2 dx + M_4 X(t)^2 \end{aligned} \quad (48)$$

$$\int_0^{s(t)} u(x, t)^2 dx \leq M_5 \int_0^{s(t)} w(x, t)^2 dx + M_6 X(t)^2 \quad (49)$$

$$\begin{aligned} \int_0^{s(t)} u_x(x,t)^2 dx &\leq 3 \int_0^{s(t)} w_x(x,t)^2 dx \\ &+ M_7 \int_0^{s(t)} w(x,t)^2 dx + M_8 X(t)^2 \end{aligned} \quad (50)$$

where

$$M_1 = 3 \left(1 + \frac{c^2 s_r^3}{3\alpha^2} \right), \quad M_2 = \frac{c^2 s_r^3}{\beta^2} \quad (51)$$

$$M_3 = 3 \frac{c^2 s_r}{\alpha^2}, \quad M_4 = 3 \frac{c^2 s_r}{\beta^2} \quad (52)$$

$$M_5 = 3 \left(1 + \frac{c s_r^2}{2\alpha} \left\{ 1 - \operatorname{sinc} \left(2\sqrt{\frac{c}{\alpha}} s_r \right) \right\} \right) \quad (53)$$

$$M_6 = \frac{3c\alpha s_r}{2\beta^2} \left\{ 1 - \operatorname{sinc} \left(2\sqrt{\frac{c}{\alpha}} s_r \right) \right\} \quad (54)$$

$$M_7 = \frac{3c^2 s_r^2}{2\alpha^2} \left\{ 1 + \operatorname{sinc} \left(2\sqrt{\frac{c}{\alpha}} s_r \right) \right\} \quad (55)$$

$$M_8 = \frac{3c^2 s_r}{2\beta^2} \left\{ 1 + \operatorname{sinc} \left(2\sqrt{\frac{c}{\alpha}} s_r \right) \right\} \quad (56)$$

Adding (47) to (48) and (49) to (50), we derive the following inequality

$$\begin{aligned} &\underline{\delta} \left(\int_0^{s(t)} u(x,t)^2 dx + \int_0^{s(t)} u_x(x,t)^2 dx + X(t)^2 \right) \\ &\leq \int_0^{s(t)} w(x,t)^2 dx + \int_0^{s(t)} w_x(x,t)^2 dx + pX(t)^2 \\ &\leq \bar{\delta} \left(\int_0^{s(t)} u(x,t)^2 dx + \int_0^{s(t)} u_x(x,t)^2 dx + X(t)^2 \right) \end{aligned} \quad (57)$$

where

$$\begin{aligned} \bar{\delta} &= \max\{M_1 + M_3, p + M_2 + M_4\} \\ \underline{\delta} &= \frac{\min\{1, p\}}{\max\{M_5 + M_7, M_6 + M_8 + 1\}} \end{aligned} \quad (58)$$

Defining the parameter $D > 0$ as

$$D = \frac{\bar{\delta}}{\underline{\delta}} e^{as_r}, \quad (59)$$

by (46) and (57), for any choice of $c > 0$ there exists $D > 0$ and $b > 0$ such that

$$\begin{aligned} &\int_0^{s(t)} u(x,t)^2 dx + \int_0^{s(t)} u_x(x,t)^2 dx + X(t)^2 \\ &\leq D \left(\int_0^{s_0} u_0(x)^2 dx + \int_0^{s_0} u'_0(x)^2 dx + X(0)^2 \right) e^{-bt}, \end{aligned} \quad (60)$$

which completes the proof of Theorem 1.

VII. NUMERICAL SIMULATION

As in [9], the simulation is performed considering a strip of zinc whose physical properties are given in Table 1 using the well known boundary immobilization method and finite difference semi-discretization. The setpoint of the interface is $s_r=0.35\text{m}$. The initial distribution of the temperature error

is set as $T_0(x) - T_m = H(s_0 - x)$ with $H=10000\text{K}\cdot\text{m}^{-1}$. The controller gain c is chosen arbitrarily, but small enough to avoid numerical instabilities, and here it is chosen $c=0.01$. The dynamics of the moving interface $s(t)$ and \mathcal{H}_1 -norm of temperature error $\|T(x,t) - T_m\|_{\mathcal{H}_1}$ are depicted in Fig. 2 and Fig. 3, respectively for two different initial position of the moving interface, namely, $s_0=0.01\text{m}$ (blue dash) and $s_0=0.25\text{m}$ (red dash). Time evolution of the control input is depicted in Fig 4. The simulation of coupled system with $s_0=0.01\text{m}$ shows that the interface converges to its setpoint while keeping $\dot{s}(t) > 0$ and $s(t) < s_r$ with a positive control signal $q_c(t) > 0$ as we expected from theoretical result, because the setpoint and initial condition satisfy (10). However, the system with the interface initialized at the position $s_0=0.25\text{m}$ leads to $\dot{s}(t) < 0$, $s(t) > s_r$, and a negative control signal because the choice of setpoint doesn't satisfy (10). Therefore, the numerical simulation is consistent with our theoretical result. We emphasize that the proposed controller does not require the restriction imposed in [9] regarding the material properties, although the equation of the controller is the same as the one proposed in [15] for a Stefan problem which describes a solidification process. In that sense, the proposed controller in the present result offers more modularity to a large class of materials, and guarantees the exponential stability of the sum of interface error and \mathcal{H}_1 -norm of temperature error, compared to [15] which provides only an asymptotical stability result.

TABLE I
PHYSICAL PROPERTIES OF ZINC

Description	Symbol	Value
Density	ρ	$6570 \text{ kg} \cdot \text{m}^{-3}$
Latent heat of fusion	ΔH^*	$111,961 \text{ J} \cdot \text{kg}^{-1}$
Heat Capacity	C_p	$389.5687 \text{ J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1}$
Thermal conductivity	k	$116 \text{ W} \cdot \text{m}^{-1}$

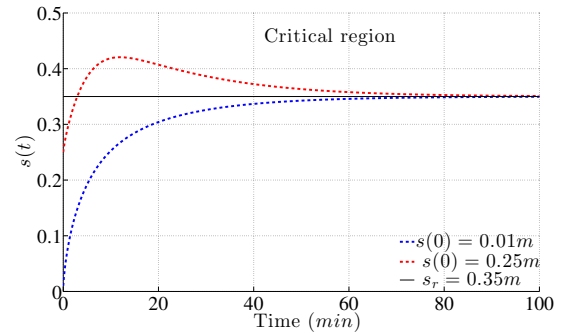


Fig. 2. The moving interface.

VIII. CONCLUSIONS AND FUTURE WORKS

Along this paper we studied a one-phase Stefan problem in 1-D and proposed a boundary feedback controller that achieves the exponential stability of sum of the moving interface and the \mathcal{H}_1 -norm of the temperature based on the

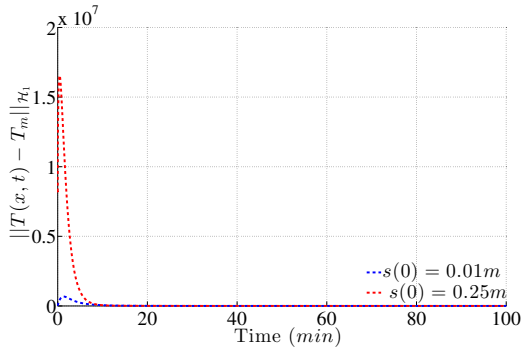


Fig. 3. \mathcal{H}_1 -norm of the temperature.

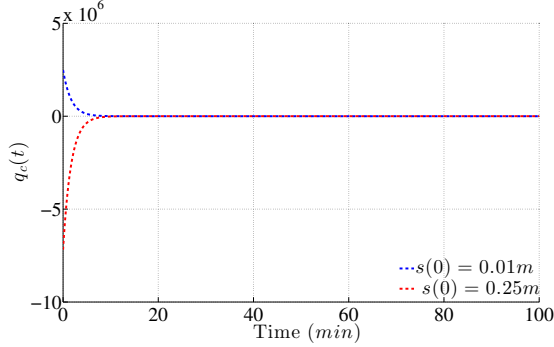


Fig. 4. The positiveness verification of the controller.

full state measurement. A nonlinear backstepping transformation for moving boundary problem is utilized and the controller is proved to remain positive, which guarantees some physical properties required for the validity of the model and the proof of stability. There are two contributions stated in our conclusion. Firstly, our approach offers an interesting perspective regarding the backstepping control of moving boundary problem whose dynamics depends on the system. Secondly, we showed the exponential stability of the sum of the interface error and the \mathcal{H}_1 -norm of temperature error, while in [9] and [15] it was shown asymptotical stability of interface error and exponential stability of \mathcal{L}_2 -norm of temperature error. Adding a dissipative term in the transformation would allow faster convergence and must be considered as a future direction. The design of an observer that enables to reconstruct the full state based on the boundary measurement is of practical interest in such type of problem and will be considered in our future work.

REFERENCES

- [1] J.S. Wettlaufer. Heat flux at the ice-ocean interface. *Journal of Geophysical Research*, 96(C4):297–313, 1991.
- [2] B. Petrus, J. Bentsman, and B.G. Thomas. Enthalpy-based feedback control algorithms for the stefan problem. In *CDC*, pages 7037–7042, 2012.
- [3] F. Conrad, D. Hilhorst, and T. I. Seidman. Well-posedness of a moving boundary problem arising in a dissolution-growth process. *Nonlinear Analysis*, 15(5):445 – 465, 1990.
- [4] B. Zalba, J.M. Marin, L.F. Cabeza, and H. Mehling. Review on thermal energy storage with phase change: materials, heat transfer analysis and applications. *Applied Thermal Engineering*, 23(3):251 – 283, 2003.
- [5] N. Daraoui, P. Dufour, H. Hammouri, and A. Hottot. Model predictive control during the primary drying stage of lyophilisation. *Control Engineering Practice*, 18(5):483–494, 2010.
- [6] A. Armaou and P.D. Christofides. Robust control of parabolic PDE systems with time-dependent spatial domains. *Automatica*, 37(1):61 – 69, 2001.
- [7] N. Petit. Control problems for one-dimensional fluids and reactive fluids with moving interfaces. In *Advances in the theory of control, signals and systems with physical modeling*, volume 407 of *Lecture notes in control and information sciences*, pages 323–337, Lausanne, Dec 2010.
- [8] Panagiotis D. Christofides. Robust control of parabolic PDE systems. *Chemical Engineering Science*, 53(16):2949 – 2965, 1998.
- [9] A. Maidi and J.-P. Corriou. Boundary geometric control of a linear stefan problem. *Journal of Process Control*, 24(6):939–946, 2014.
- [10] C. Karvaris and J.C. Kantor. Geometric methods for nonlinear process control i. *Background*, *Industrial & Engineering Chemistry Research*, 29:2295–2310, 1990.
- [11] C. Karvaris and J.C. Kantor. Geometric methods for nonlinear process control ii. *Controller synthesis*, *Industrial & Engineering Chemistry Research*, 29:2310–2323, 1990.
- [12] A. Maidi, M. Diaf, and J.-P. Corriou. Boundary geometric control of a counter-current heat exchanger. *Journal of Process Control*, 19(2):297–313, 2009.
- [13] M. Krstic and A. Smyshlyaev. *Boundary control of PDEs: A course on backstepping designs*, volume 16. Siam, 2008.
- [14] A. Smyshlyaev and M. Krstic. Closed-form boundary state feedbacks for a class of 1-d partial integro-differential equations. *Automatic Control, IEEE Transactions on*, 49(12):2185–2202, Dec 2004.
- [15] Bryan Petrus, Joseph Bentsman, and Brian G Thomas. Feedback control of the two-phase stefan problem, with an application to the continuous casting of steel. In *Decision and Control (CDC), 2010 49th IEEE Conference on*, pages 1731–1736. IEEE, 2010.
- [16] M. Izadi and S. Dubljevic. Backstepping output-feedback control of moving boundary parabolic PDEs. *European Journal of Control*, 21(0):27 – 35, 2015.
- [17] M. Krstic. Compensating actuator and sensor dynamics governed by diffusion PDEs. *Systems & Control Letters*, 58(5):372–377, 2009.
- [18] G.A. Susto and M. Krstic. Control of PDE-ODE cascades with neu-mann interconnections. *Journal of the Franklin Institute*, 347(1):284–314, 2010.
- [19] S. Tang and C. Xie. State and output feedback boundary control for a coupled PDE-ODE system. *Systems & Control Letters*, 60(8):540–545, 2011.
- [20] S. Gupta. *The classical Stefan problem. Basic concepts, Modelling and Analysis*. Applied mathematics and Mechanics. North-Holland, 2003.